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Classification of irreducible integrable representations for the full toroidal lie algebras☆

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Abstract

The purpose of this paper is to classify irreducible integrable modules of the full toroidal Lie-algebras, with finite-dimensional weight spaces and non-zero central charge.

MSC: 17B65; 17B67; 17B68

1. Introduction

It is well known that the representation theory of Virasoro algebra and affine Kac–Moody Lie-algebra plays an important role both in physics and in mathematics. For example see the book [8] and [10].

The Virasoro algebra acts on any (except when the level is negative of dual coxeter number) highest weight module of the affine Lie-algebra through the use of famous Sugawara operators. It is well known that affine Lie-algebras admit representation on the Fock space and hence admits representation of the Virasoro algebra. Thus the semidirect product of Virasoro algebra and affine Kac–Moody Lie-algebra with common center is an interesting object to study. The generalization of this classical object is the subject of the current paper.

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The affine Kac–Moody Lie-algebra is the universal central extension of algebraic maps from one-dimensional torus to finite-dimensional simple Lie-algebra $\mathscr{G}$. The Virasoro Lie-algebra is the universal central extension of Lie algebra of diffeomorphisms of one-dimensional torus. Both algebras can be defined in an algebraic way as the universal central extension of $\mathscr{G} \otimes \mathbb{C}[t, t^{-1}]$ for affine Lie-algebra and the universal central extension of Der $\mathbb{C}[t, t^{-1}]$ for Virasoro algebra.

The generalization of the affine Kac–Moody Lie-algebra is the so-called toroidal Lie-algebra which can be defined as the universal central extension of $\mathscr{G} \otimes \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$. To generalize Virasoro algebra we have to first note that Der $\mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ is centrally closed for $\nu \geqslant 1$ [12]. Nevertheless Der $\mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ has an interesting abelian extension, which has emerged while generalizing the vertex operator construction on the Fock space in [16]. It is interesting to note that the abelian part is precisely the center of the toroidal Lie-algebra. Thus the semi-direct product of Der $\mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ and $\mathscr{G} \otimes \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ with common extension which is denoted by τ and called full toroidal Lie-algebra has emerged as an interesting mathematical object.

The first important question is to construct a representation for τ through known methods. Several attempts have been made in [16,4,5]. Eventually in a remarkable paper [6] Billig succeeded in constructing a class representations for τ using vertex operator algebras. These modules are irreducible integrable and have finite-dimensional weight spaces.

The purpose of this paper is to classify irreducible integrable modules for τ with finite-dimensional weight spaces. The proof of our result depends heavily on [14,15,9]. In [9] the classification of irreducible integrable modules for a certain proper subalgebra has been attempted and the classification problem has been reduced to the classification of $(A, \operatorname{Der} A)$ modules where $A = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$. In the present work using similar methods we have reduced the problem for τ to the problem for $(A, \operatorname{Der} A)$ modules. The classification of irreducible $(A, \operatorname{Der} A)$ modules with finite-dimensional weight spaces is now available in [15]. Putting the above results together we have classified irreducible integrable modules for τ in Theorem 3.3 for the non-zero center case. The zero center case has been given in Theorem 4.3.

2. Basic concepts and results

Let $\dot{\mathfrak{g}}$ denote a finite-dimensional simple Lie-algebra over $\mathbb{C}$, $\dot{\mathfrak{h}}$ a Cartan subalgebra, $\dot{\Delta}$ the root system of $\dot{\mathfrak{g}}$, and $\dot{\Delta}_+ (\dot{\Delta}_-)$ the set of positive roots (negative roots). Then $\dot{\mathfrak{g}} = \dot{\mathfrak{h}} \bigoplus_{\alpha \in \dot{\Delta}} \dot{\mathfrak{g}}_\alpha$. For $\alpha \in \dot{\Delta}$, let $\alpha^\vee \in \dot{\mathfrak{h}}$ be such that $\alpha(\alpha^\vee) = 2$. Let $e_\alpha \in \dot{\mathfrak{g}}_\alpha, e_{-\alpha} \in \dot{\mathfrak{g}}_{-\alpha}$ be such that $[e_\alpha, e_{-\alpha}] = \alpha^\vee$, $[\alpha^\vee, e_\alpha] = 2e_\alpha$, $[\alpha^\vee, e_{-\alpha}] = -2e_{-\alpha}$. Let

$$\dot{\mathfrak{g}}_+ = \bigoplus_{\alpha \in \dot{\Delta}_+} \dot{\mathfrak{g}}_\alpha, \quad \dot{\mathfrak{g}}_- = \bigoplus_{\alpha \in \dot{\Delta}_-} \dot{\mathfrak{g}}_\alpha, \quad \dot{Q}_+ = \sum_{i=1}^{l} \mathbb{Z}_+ \alpha_i, \quad \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.$$

Let $A = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}] (\nu \geqslant 1)$ be the ring of Laurent polynomials in commuting variables $t_0, t_1, \ldots, t_\nu$. For $\mathbf{n} = (n_1, n_2, \ldots, n_\nu) \in \mathbb{Z}^\nu$, $n_0 \in \mathbb{Z}$, we denote $t_0^{n_0} t_1^{n_1} \cdots t_\nu^{n_\nu}$ by

$t_0^{n_0}t^{\mathbf{n}}$. Let $\tilde{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes A$ be the tensor product of $\dot{\mathfrak{g}}$ and A with the Lie bracket:

$$[x_1 \otimes f_1, x_2 \otimes f_2] = [x_1, x_2] \otimes f_1 f_2,$$

where $x_1, x_2 \in \dot{\mathfrak{g}}$, f_1, $f_2 \in A$. Then $\tilde{\mathfrak{g}}$ can be viewed as the algebra of $\dot{\mathfrak{g}}$-valued polynomial functions on a torus (see [13] for discussion of integrable modules of $\tilde{\mathfrak{g}}$). Let $\tilde{\mathscr{K}}$ be the free $\mathscr{A}$-module with basis $\{k_0, k_1, \ldots, k_v\}$ and let $d\tilde{\mathscr{K}}$ be the subspace spanned by all elements of the form $\sum_{i=0}^{v} r_i t_0^{r_0} t^{\mathbf{r}} k_i$, for $(r_0, \mathbf{r}) = (r_0, r_1, \ldots, r_v) \in \mathbb{Z}^{v+1}$. Let $\mathscr{K} = \tilde{\mathscr{K}}/d\tilde{\mathscr{K}}$ and denote the image of $t_0^{r_0}t^{\mathbf{r}}k_i$ still by itself. Then $\mathscr{K}$ is spanned by the elements $\{t_0^{r_0}t^{\mathbf{r}}k_p | p = 0, 1, \ldots, v, r_0 \in \mathbb{Z}, \mathbf{r} \in \mathbb{Z}^v\}$with the following relations:

$$\sum_{p=0}^{v} r_p t_0^{r_0} t^{\mathbf{r}} k_p = 0. \tag{2.1}$$

The toroidal Lie algebra associated to $\dot{\mathfrak{g}}$ is

$$\hat{\tau} = \dot{\mathfrak{g}} \otimes A \oplus \mathscr{K}$$

with the bracket:

$$[g_1 \otimes f_1, g_2 \otimes f_2] = [g_1, g_2] \otimes f_1 f_2 + (g_1|g_2) \sum_{p=0}^{v} (d_p(f_1) f_2) k_p \tag{2.2}$$

and

$$[\hat{\tau}, \mathscr{K}] = 0, \tag{2.3}$$

$(\cdot|\cdot)$ is the normal invariant symmetric bilinear form on $\dot{\mathfrak{g}}$ [11]. d_p is the degree derivation of A, i.e.,

$$d_p = t_p \frac{\mathrm{d}}{\mathrm{d}t_p}, \quad p = 0, 1, \ldots, v.$$

Let $\mathscr{D}$ be the Lie-algebra of derivations on A. Then

$$\mathscr{D} = \left\{ \sum_{p=0}^{v} f_p(t_0, t_1, \ldots, t_v) d_p | f_p(t_0, t_1, \ldots, t_v) \in A \right\}.$$

For $D \in \mathscr{D}$, D can be naturally extended to a derivation on the tensor product $\dot{\mathfrak{g}} \otimes A$ by [3]

$$D(x \otimes f) = x \otimes Df, \quad x \in \dot{\mathfrak{g}}, \ f \in A$$

and D has a unique extension to the universal covering algebra $\hat{\tau}$ of $\dot{\mathfrak{g}} \otimes A$ by

$$t_0^{m_0} t^{\mathbf{m}} d_a (t_0^{n_0} t^{\mathbf{n}} k_b) = n_a t_0^{m_0+n_0} t^{\mathbf{m}+\mathbf{n}} k_b + \delta_{ab} \sum_{p=0}^{v} m_p t_0^{m_0+n_0} t^{\mathbf{m}+\mathbf{n}} k_p.$$

It is known that the algebra $\mathscr{D}$ admits two non-trivial 2-cocyles with values in $\mathscr{K}$ (see [4]):

$$\phi_1(t_0^{m_0}t^{\mathbf{m}}d_a, t_0^{n_0}t^{\mathbf{n}}d_b) = -n_a m_b \sum_{p=0}^{\nu} m_p t_0^{m_0+n_0} t^{\mathbf{m}+\mathbf{n}} k_p,$$

$$\phi_2(t_0^{m_0}t^{\mathbf{m}}d_a, t_0^{n_0}t^{\mathbf{n}}d_b) = m_a n_b \sum_{p=0}^{\nu} m_p t_0^{m_0+n_0} t^{\mathbf{m}+\mathbf{n}} k_p.$$

Let ϕ be an arbitrary linear combination of ϕ_1 and ϕ_2. Then there is a corresponding Lie-algebra

$$\tau = \dot{\mathfrak{g}} \otimes A \oplus \mathscr{K} \oplus \mathscr{D}$$

with the Lie bracket (2.1)–(2.2) and the following:

$$[t_0^{m_0}t^{\mathbf{m}}d_a, t_0^{n_0}t^{\mathbf{n}}k_b] = n_a t_0^{m_0+n_0}t^{\mathbf{m}+\mathbf{n}}k_b + \delta_{ab} \sum_{p=0}^{\nu} m_p t_0^{m_0+n_0} t^{\mathbf{m}+\mathbf{n}} k_p, \tag{2.4}$$

$$\begin{aligned}[t_0^{m_0}t^{\mathbf{m}}d_a, t_0^{n_0}t^{\mathbf{n}}d_b] = {} & n_a t_0^{m_0+n_0}t^{\mathbf{m}+\mathbf{n}}d_b - m_b t_0^{m_0+n_0}t^{\mathbf{m}+\mathbf{n}}d_a \\ & + \phi(t_0^{m_0}t^{\mathbf{m}}d_a, t_0^{n_0}t^{\mathbf{n}}d_b),\end{aligned} \tag{2.5}$$

$$[t_0^{m_0}t^{\mathbf{m}}d_a, x \otimes t_0^{n_0}t^{\mathbf{n}}] = n_a x \otimes t_0^{m_0+n_0}t^{\mathbf{m}+\mathbf{n}}. \tag{2.6}$$

We call τ the full toroidal Lie-algebra associated to $\dot{\mathfrak{g}}$ and ϕ. Let

$$\mathfrak{h} = \dot{\mathfrak{h}} \oplus \left(\bigoplus_{i=0}^{\nu} \mathbb{C}k_i \right) \oplus \left(\bigoplus_{i=0}^{\nu} \mathbb{C}d_i \right). \tag{2.7}$$

Then $\mathfrak{h}$ is an abelian Lie subalgebra of τ. Let $\delta_i, \Lambda_i \in \mathfrak{h}^*(i = 0, 1, \ldots, \nu)$ be such that

$$\Lambda_i(\dot{\mathfrak{h}}) = 0, \quad \Lambda_i(k_j) = \delta_{ij}, \quad \Lambda_i(d_j) = 0, \quad i, j = 0, 1, \ldots, \nu, \tag{2.8}$$

$$\delta_i(\dot{\mathfrak{h}}) = 0, \quad \delta_i(k_j) = 0, \quad \delta_i(d_j) = \delta_{ij}, \quad i, j = 0, 1, \ldots, \nu \tag{2.9}$$

and denote $\sum_{i=1}^{\nu} m_i \delta_i$ by $\delta_{\mathbf{m}}$, $\mathbf{m} = (m_1, m_2, \ldots, m_\nu) \in \mathbb{Z}^\nu$. Then τ has the root space decomposition with respect to $\mathfrak{h}$ as follows:

$$\tau = \mathfrak{h} \oplus \left(\bigoplus_{\beta \in \Delta} \tau_\beta \right),$$

where $\Delta = \dot{\Delta} \cup \{\alpha + m_0\delta_0 + \delta_{\mathbf{m}} | \alpha \in \dot{\Delta} \cup \{0\}, \mathbf{m} \in \mathbb{Z}^\nu, m_0 \in \mathbb{Z}, (m_0, \mathbf{m}) \neq (0, \mathbf{0})\}$ and

$$\tau_{\alpha+m_0\delta_0+\delta_{\mathbf{m}}} = \dot{\mathfrak{g}}_\alpha \otimes t_0^{m_0}t^{\mathbf{m}},$$

$$\tau_{m_0\delta_0+\delta_{\mathbf{m}}} = \dot{\mathfrak{h}} \otimes t_0^{m_0}t^{\mathbf{m}} \oplus \left(\bigoplus_{i=0}^{\nu} \mathbb{C}t_0^{m_0}t^{\mathbf{m}}k_i \right) \oplus \left(\bigoplus_{i=0}^{\nu} \mathbb{C}t_0^{m_0}t^{\mathbf{m}}d_i \right).$$

Let

$$\mathfrak{b} = \mathscr{K} \oplus \mathscr{D}$$

and

$$\mathfrak{b}_+ = \sum_{p=0}^{v} t_0\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}]k_p \oplus \sum_{p=0}^{v} t_0\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}]d_p,$$

$$\mathfrak{b}_- = \sum_{p=0}^{v} t_0^{-1}\mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}]k_p \oplus \sum_{p=0}^{v} t_0^{-1}\mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}]d_p,$$

$$\mathfrak{b}_0 = \sum_{p=0}^{v} \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}]k_p \oplus \sum_{p=0}^{v} \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}]d_p,$$

$$\tau_+ = \dot{\mathfrak{g}}_+ \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus \dot{\mathfrak{g}} \otimes t_0\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus \mathfrak{b}_+,$$

$$\tau_- = \dot{\mathfrak{g}}_- \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus \dot{\mathfrak{g}} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus \mathfrak{b}_-,$$

$$\tau_0 = \dot{\mathfrak{h}} \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus \mathfrak{b}_0.$$

Then

$$\mathfrak{b} = \mathfrak{b}_+ \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_-,$$
$$\tau = \tau_+ \oplus \tau_0 \oplus \tau_-.$$

Extend $\alpha \in \dot{\Delta}$ to the element in $\mathfrak{h}^*$ by $\alpha(k_i) = \alpha(d_i) = 0$ $(0 \leqslant i \leqslant v)$ and the normal non-degenerate symmetric bilinear form $(\cdot|\cdot)$ on $\dot{\mathfrak{h}}^*$ to a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ by

$$(\alpha_i|\delta_k) = (\alpha_i|\Lambda_k) = 0, \quad 1 \leqslant i, \quad j \leqslant 1,$$

$$(\delta_k|\delta_p) = (\Lambda_k|\Lambda_p) = 0, \quad (\delta_k|\Lambda_p) = \delta_{kp}, \quad 0 \leqslant k, \quad p \leqslant v.$$

For $\gamma = \alpha + m_0\delta_0 + \delta_{\mathbf{m}} \in \Delta$, where $\alpha \in \dot{\Delta}$, γ is called a real root, if $(\gamma|\gamma) \neq 0$. Denote the set of all real roots by Δ^{re}. Define

$$\gamma^\vee = \alpha^\vee + \frac{2}{(\alpha|\alpha)} \sum_{i=0}^{v} m_i k_i.$$

Then

$$\gamma(\gamma^\vee) = \alpha(\alpha^\vee) = 2.$$

Let γ be a real root. Define reflection on $\mathfrak{h}^*$ by

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma, \quad \lambda \in \mathfrak{h}^*.$$

Let $\mathscr{W}$ be the Weyl group generated by $\{r_\gamma | \gamma \in \Delta^{\mathrm{re}}\}$. Then $(\cdot|\cdot)$ defined above is $\mathscr{W}$-invariant. See [1,2] for some interesting results on Weyl groups in the context of Toroidal Lie-algebras (or more generally the Extended Affine Lie-algebras).

Definition 2.1. A module V of τ is called integrable if

(1) V admits a weight space decomposition , i.e.,

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where $V_\lambda = \{v \in V | h.v = \lambda(h)v, \forall h \in \mathfrak{h}\}$. Denote by $P(V)$ the set of all weights.

(2) For $\alpha \in \Delta$, $m_0 \in \mathbb{Z}$, $\mathbf{m} \in \mathbb{Z}^\nu$, $e_\alpha \otimes t_0^{m_0} t^{\mathbf{m}}$ is locally nilpotent on V.

Let ϑ_{fin} be the category of irreducible integrable τ-modules with finite-dimensional weight spaces. Similar to the proof of Lemma 2.3 in [14], we can obtain the following results (one can also see [7]):

Lemma 2.1. Let $V \in \vartheta_{\text{fin}}$. Then

(1) *$P(V)$ is $\mathscr{W}$-invariant.*
(2) dim $V_\lambda =$ dim $V_{\omega\lambda}$, $\omega \in \mathscr{W}$, $\lambda \in P(V)$.
(3) *For $\alpha \in \Delta^{\text{re}}$, $\lambda \in P(V)$, we have $\lambda(\alpha^\vee) \in \mathbb{Z}$.*
(4) *Let $\alpha \in \Delta^{\text{re}}$, $\lambda \in P(V)$. If $\lambda(\alpha^\vee) > 0$, then $\lambda - \alpha \in P(V)$.*
(5) *For $\lambda \in P(V)$, $\lambda(k_i)$ is a constant integer, $i = 0, 1, \ldots, \nu$.*

By Lemma 2.1, we can assume that

$$\lambda(k_i) = c_i, \quad i = 0, 1, \ldots, \nu, \quad \forall \lambda \in P(V), \quad c_i \in \mathbf{Z}. \tag{2.10}$$

Throughout the paper, $c_i (i = 0, 1, \ldots, \nu)$ are always defined by (2.10). For $\underline{m} = (m_0, \mathbf{m})$, denote $t_0^{m_0} t^{\mathbf{m}}$ by $t^{\underline{m}}$.

Lemma 2.2. Let $A = (a_{ij})(0 \leqslant i, j \leqslant \nu)$ be a $(\nu + 1) \times (\nu + 1)$*-matrix such that* det $A = 1$ *and $a_{ij} \in \mathbb{Z}$. Then there exists an automorphism σ of τ such that*

$$\sigma(x \otimes t^{\underline{m}}) = x \otimes t^{\underline{m}A^{\mathrm{T}}},$$

$$\sigma(t^{\underline{m}} k_j) = \sum_{p=0}^{\nu} a_{pj} t^{\underline{m}A^{\mathrm{T}}} k_p, \quad 0 \leqslant j \leqslant \nu,$$

$$\sigma(t^{\underline{m}} d_j) = \sum_{p=0}^{\nu} b_{jp} t^{\underline{m}A^{\mathrm{T}}} d_p, \quad 0 \leqslant j \leqslant \nu,$$

where $B = (b_{ij}) = A^{-1}$.

Let $V \in \vartheta_{\text{fin}}$. If V has non-zero central charges, it follows from Lemma 2.2 that we can assume that $c_0 \neq 0$, $c_1 = \cdots = c_\nu = 0$. Similar to the proof of Theorem 2.1 in [9], we have

Theorem 2.1. *Let $V \in \vartheta_{\text{fin}}$. Then*

(1) *If $c_0 > 0$ and $c_1 = c_2 = \cdots = c_\nu = 0$, then*

$$\{v \in V|_{\tau_+} \cdot v = 0\} \neq 0.$$

(2) *If $c_0 < 0$ and $c_1 = c_2 = \cdots = c_\nu = 0$, then*

$$\{v \in V|_{\tau_-} \cdot v = 0\} \neq 0.$$

(3) *If $c_0 = c_1 = \cdots = c_\nu = 0$, then there exist non-zero elements $v, w \in V$ such that*

$$(\dot{\mathfrak{g}}_+ \otimes A) \cdot v = 0, \quad (\dot{\mathfrak{g}}_- \otimes A) \cdot w = 0.$$

3. Modules of τ in ϑ_{fin} with non-zero central charges

In this section, we discuss the structure of $V \in \vartheta_{\text{fin}}$ which has non-zero central charges. By Lemma 2.2, we can assume that $c_0 \neq 0$, $c_1 = c_2 = \cdots = c_\nu = 0$. Let

$$T = \{v \in V | \tau_+ \cdot v = 0\} \text{ if } c_0 > 0 \quad \text{or} \quad T = \{v \in V | \tau_- \cdot v = 0\} \text{ if } c_0 < 0.$$

Obviously, T is a τ_0-module. Since V is irreducible, it follows that

$$V = U(\tau_-) \cdot T \quad \text{or} \quad V = U(\tau_+) \cdot T.$$

Therefore T is irreducible as a τ_0-module. Let

$$T = \bigoplus_{\mathbf{m} \in \mathbb{Z}^\nu} T_{\mathbf{m}}, \tag{3.1}$$

where $T_{\mathbf{m}} = \{v \in T | d_i(v) = (\lambda_0(d_i) + m_i)v\}$, for a fixed $\lambda_0 \in P(V)$, $\mathbf{m} = (m_1, m_2, \ldots, m_\nu) \in \mathbb{Z}^\nu$. Then T is $\mathbb{Z}^\nu$-graded. By Theorem 2.1 and the fact that V has finite-dimensional weight spaces, $T_{\mathbf{m}}$ is finite dimensional.

Let

$$\mathfrak{g}_a = \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm 1}] \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0.$$

Then $\mathfrak{g}_a$ is an affine Lie subalgebra of τ. Let

$$\mathfrak{h}_a = \dot{\mathfrak{h}} \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0.$$

Similar to the proof in [9], we can deduce that

Lemma 3.1. *For any $\mathbf{m} \in \mathbb{Z}^\nu$, $v \in T \backslash \{0\}$, we have $t^{\mathbf{m}} k_0 v \neq 0$, and*

$$\dim T_{\mathbf{n}} = \dim T_{\mathbf{m}} = n$$

for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^\nu$.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of T_0. Set

$$v_i(\mathbf{m}) = \frac{1}{c_0} t^{\mathbf{m}} k_0 \cdot v_i, \quad i = 1, 2, \ldots, n. \tag{3.2}$$

Then $\{v_1(\mathbf{m}), v_2(\mathbf{m}), \ldots, v_n(\mathbf{m})\}$ is a basis of $T_{\mathbf{m}}$. Assume that

$$\begin{aligned} &\frac{1}{c_0} t^{\mathbf{m}} k_0(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_n(\mathbf{n})) \\ &\quad = (v_1(\mathbf{m}+\mathbf{n}), v_2(\mathbf{m}+\mathbf{n}), \ldots, v_n(\mathbf{m}+\mathbf{n})) B_{\mathbf{m},\mathbf{n}}. \end{aligned} \tag{3.3}$$

By Lemma 3.1, $B_{\mathbf{m},\mathbf{n}}$ is a $n \times n$ invertible matrix and

$$B_{\mathbf{m},\mathbf{n}} B_{\mathbf{r},\mathbf{s}} = B_{\mathbf{r},\mathbf{s}} B_{\mathbf{m},\mathbf{n}}, \quad B_{\mathbf{m},\mathbf{n}} = B_{\mathbf{n},\mathbf{m}}.$$

We can assume that $\{B_{\mathbf{m},\mathbf{n}} | \mathbf{m}, \mathbf{n} \in \mathbb{Z}^v\}$ are all upper triangular matrices.

Lemma 3.2. *For any* $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^v$, $B_{\mathbf{m},\mathbf{n}} - I$ *is a strictly upper triangular matrix.*

Proof. It is similar to the proof of Lemmas 3.5–3.9. □

For $\mathbf{m} = (m_1, \ldots, m_v)$, $\mathbf{n} = (n_1, \ldots, n_v) \in \mathbb{Z}^v$, assume that

$$t^{\mathbf{m}} d_a(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_n(\mathbf{n})) = (v_1(\mathbf{m}+\mathbf{n}), v_2(\mathbf{m}+\mathbf{n}), \ldots, v_n(\mathbf{m}+\mathbf{n})) A^{(a)}_{\mathbf{m},\mathbf{n}},$$

where $A^{(a)}_{\mathbf{m},\mathbf{n}} \in \mathbb{C}^{n \times n}$. Since $[t^{\mathbf{m}} d_a, t^{\mathbf{n}} k_0] = n_a t^{\mathbf{m}+\mathbf{n}} k_0$, $1 \leqslant a \leqslant v$, we have

$$A^{(a)}_{\mathbf{m},\mathbf{n}} = B_{\mathbf{m},\mathbf{n}} A^{(a)}_{\mathbf{m},\underline{\mathbf{0}}} + n_a I, \quad 1 \leqslant a \leqslant v. \tag{3.4}$$

Theorem 3.1. *For all* $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^v$, $h \in \mathfrak{h}_a$ *and* $p = 1, 2, \ldots, v$, *we have*

$$t^{\mathbf{m}} k_p \cdot T = 0, \tag{3.5}$$

$$t^{\mathbf{m}} k_0 t^{\mathbf{n}} k_0 = c_0 t^{\mathbf{m}+\mathbf{n}} k_0, \quad B_{\mathbf{m},\mathbf{n}} = I, \tag{3.6}$$

$$\begin{aligned} &h \otimes t^{\mathbf{m}}(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_n(\mathbf{n})) \\ &\quad = \Lambda(h)(v_1(\mathbf{m}+\mathbf{n}), v_2(\mathbf{m}+\mathbf{n}), \ldots, v_n(\mathbf{m}+\mathbf{n})), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &t^{\mathbf{m}} d_0(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_n(\mathbf{n})) \\ &\quad = \Lambda(d_0)(v_1(\mathbf{m}+\mathbf{n}), v_2(\mathbf{m}+\mathbf{n}), \ldots, v_n(\mathbf{m}+\mathbf{n})), \end{aligned} \tag{3.8}$$

where $\Lambda \in P(V)$.

Proof. For $\mathbf{m} = (m_1, m_2, \ldots, m_v) \in \mathbb{Z}^v$. Assume that

$$t^{\mathbf{m}} k_p(v_1, v_2, \ldots, v_n) = (v_1(\mathbf{m}), v_2(\mathbf{m}), \ldots, v_n(\mathbf{m})) C_{\mathbf{m},p}.$$

Then

$$\begin{aligned}
&t^{\mathbf{n}}k_q t^{\mathbf{m}}k_p(v_1, v_2, \ldots, v_n)\\
&\quad = t^{\mathbf{n}}k_q(v_1(\mathbf{m}), v_2(\mathbf{m}), \ldots, v_n(\mathbf{m}))C_{\mathbf{m},p}\\
&\quad = \frac{1}{c_0}t^{\mathbf{m}}k_0 t^{\mathbf{n}}k_q(v_1, v_2, \ldots, v_n)C_{\mathbf{m},p}\\
&\quad = \frac{1}{c_0}t^{\mathbf{m}}k_0(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_n(\mathbf{n}))C_{\mathbf{n},q}C_{\mathbf{m},p}\\
&\quad = \frac{1}{c_0}t^{\mathbf{m}}k_0\frac{1}{c_0}t^{\mathbf{n}}k_0(v_1, v_2, \ldots, v_n)C_{\mathbf{n},q}C_{\mathbf{m},p} = t^{\mathbf{m}}k_p t^{\mathbf{n}}k_q(v_1, v_2, \ldots, v_n)\\
&\quad = \frac{1}{c_0}t^{\mathbf{n}}k_0\frac{1}{c_0}t^{\mathbf{m}}k_0(v_1, v_2, \ldots, v_n)C_{\mathbf{m},p}C_{\mathbf{n},p}.
\end{aligned}$$

Therefore

$$C_{\mathbf{n},q}C_{\mathbf{m},p} = C_{\mathbf{m},p}C_{\mathbf{n},q}.$$

This means that the Lie-algebra $\mathscr{C}$ spanned by $\{C_{\mathbf{m},p} | \mathbf{m} \in \mathbb{Z}^\nu, 1 \leqslant p \leqslant \nu\}$ is a commutative Lie-algebra. Let $\mathbf{m} = (m_1, m_2, \ldots, m_\nu) \in \mathbb{Z}^\nu$ be such that $m_a \neq 0$, for some $1 \leqslant a \leqslant \nu$. Since

$$[t^{-\mathbf{m}}d_a, t^{\mathbf{m}}k_p] = 0,$$

it follows that

$$(A^{(a)}_{-\mathbf{m},\mathbf{0}} + m_a B^{-1}_{-\mathbf{m},\mathbf{m}})C_{\mathbf{m},p} = C_{\mathbf{m},p}A^{(a)}_{-\mathbf{m},\mathbf{0}}.$$

If $C_{\mathbf{m},p}$ is invertible, then $A^{(a)}_{-\mathbf{m},\mathbf{0}}$ and $A^{(a)}_{-\mathbf{m},\mathbf{0}} + m_a B^{-1}_{-\mathbf{m},\mathbf{m}}$ are similar matrices, which is impossible, since m_a is non-zero and $B^{-1}_{-\mathbf{m},\mathbf{m}}$ is an upper triangular matrix with elements on the diagonal line being all one. We deduce that $t^{\mathbf{m}}k_p$ is locally nilpotent. Note that

$$(t^{\mathbf{m}}k_p)^s(v_1, v_2, \ldots, v_n) = \left(\frac{1}{c_0}t^{\mathbf{m}}k_0\right)^s(v_1, v_2, \ldots, v_n)C^s_{\mathbf{m},p},$$

so $C_{\mathbf{m},\mathbf{p}}$ is a nilpotent matrix. Since $\mathscr{C}$ is commutative, there exists a non-zero element v in T such that

$$t^{\mathbf{m}}k_p(v) = 0, \quad \forall \mathbf{m} \in \mathbb{Z}^\nu,\ 1 \leqslant p \leqslant \nu.$$

Let $T_1 = \{v \in T | t^{\mathbf{m}}k_p(v) = 0, \forall \mathbf{m} \in \mathbb{Z}^\nu, 1 \leqslant p \leqslant \nu\}$. Then T_1 is a non-zero submodule of τ_0-module T and (3.5) follows from the fact that T is irreducible. The proof of (3.6)–(3.8) is similar to the proof of Theorem 3.1 in [9]. $\square$

By Theorem 3.1, T is an irreducible $\mathfrak{b}_0$-module. Let $A_\nu = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_\nu^{\pm 1}]$, and Der A_ν the derivation algebra of A_ν with the following Lie bracket:

$$[t^{\mathbf{m}}d_a, t^{\mathbf{n}}d_b] = n_a t^{\mathbf{m}+\mathbf{n}}d_b - m_b t^{\mathbf{m}+\mathbf{n}}d_a.$$

Extending the above Lie bracket to $A_\nu \oplus \mathrm{Der}\, A_\nu$ by

$$[t^{\mathbf{m}}, t^{\mathbf{n}}] = 0, \quad [t^{\mathbf{m}} d_a, t^{\mathbf{n}}] = n_a t^{\mathbf{m}+\mathbf{n}}.$$

Then $A_\nu \oplus \mathrm{Der}\, A_\nu$ is a Lie-algebra. Now define the action of A_ν on T by

$$t^{\mathbf{m}}(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_n(\mathbf{n})) \\ = (v_1(\mathbf{m}+\mathbf{n}), v_2(\mathbf{m}+\mathbf{n}), \ldots, v_n(\mathbf{m}+\mathbf{n})), \quad \forall \mathbf{m}, \mathbf{n} \in \mathbb{Z}^\nu.$$

It is easy to see from (3.4) that T is an irreducible $\mathbb{Z}^\nu$-graded $A_\nu \oplus \mathrm{Der}\, A_\nu$-module with finite-dimensional weight spaces.

Theorem 3.2 (*Rao [15]*). *As $A_\nu \oplus \mathrm{Der}\, A_\nu$-module, T is isomorphic to $F^\alpha(\psi, b)$, for some (α, ψ, b), where $\alpha = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{C}^\nu$, $b \in \mathbb{C}$, $F^\alpha(\psi, b) = V(\psi, b) \otimes A_\nu$ is an irreducible $A_\nu \oplus \mathrm{Der}\, A_\nu$-module such that $V(\psi, b)$ is a n-dimensional irreducible $\mathrm{gl}_\nu(\mathbb{C})$-module, and*

$$\psi(I) = b\, \mathrm{id}_{V(\psi,b)},$$

$$t^{\mathbf{r}} d_p(w \otimes t^{\mathbf{m}}) = (m_p + \alpha_p) w \otimes t^{\mathbf{r}+\mathbf{m}} + \sum_{i=1}^{\nu} r_i \psi(E_{ip}) w \otimes t^{\mathbf{r}+\mathbf{m}},$$

$$t^{\mathbf{m}}(w \otimes t^{\mathbf{n}}) = w \otimes t^{\mathbf{m}+\mathbf{n}},$$

where $w \in V(\psi, b)$. Therefore T is isomorphic to$F^\alpha(\psi, b)$, for some (α, ψ, b) as Der A_ν-*modules.*

Let $F^\alpha(\psi, b)$ be an irreducible module of $A_\nu \oplus \mathrm{Der}\, A_\nu$ defined as in Theorem 3.2, and $\Lambda \in \mathfrak{h}_a^*$ be such that $\Lambda(k_0) = c_0 \neq 0$. Extend $F^\alpha(\psi, b)$ to be a τ_0-module as follows:

$$t^{\mathbf{r}} k_0(w \otimes t^{\mathbf{m}}) = c_0 w \otimes t^{\mathbf{r}+\mathbf{m}}, \quad t^{\mathbf{r}} k_p(w \otimes t^{\mathbf{m}}) = 0, \quad 1 \leqslant p \leqslant \nu,$$

$$h \otimes t^{\mathbf{r}}(w \otimes t^{\mathbf{m}}) = \Lambda(h) w \otimes t^{\mathbf{r}+\mathbf{m}}, \quad t^{\mathbf{r}} d_0(w \otimes t^{\mathbf{m}}) = \Lambda(d_0) w \otimes t^{\mathbf{r}+\mathbf{m}},$$

where $h \in \dot{\mathfrak{h}}$. Denote $F^\alpha(\psi, b)$ by $F^\alpha(\psi, b, \Lambda)$ as the module of τ_0. Let τ_+ (if $c_0 > 0$) or τ_- (if $c_0 < 0$) act on $F^\alpha(\psi, b, \Lambda)$ by zero and so we have the induced module for τ:

$$F_\tau^\alpha(\psi, b, \Lambda) = \mathrm{Ind}_{\tau_+ + \tau_0}^{\tau}(F^\alpha(\psi, b, \Lambda)) \quad \text{if } c_0 > 0,$$

or

$$F_\tau^\alpha(\psi, b, \Lambda) = \mathrm{Ind}_{\tau_- + \tau_0}^{\tau}(F^\alpha(\psi, b, \Lambda)) \quad \text{if } c_0 < 0.$$

Theorem 3.3. *Let $F_\tau^\alpha(\psi, b, \Lambda)(\Lambda(k_0) = c_0 \neq 0)$ be defined as above. Then*

(1) *There is a maximal one among the submodules of $F_\tau^\alpha(\psi, b, \Lambda)$ intersecting $F^\alpha(\psi, b, \Lambda)$ trivially. We denote the maximal submodule by $F_\tau^\alpha(\psi, b, \Lambda)'$.*

(2) *$\bar{F}^{\alpha}_{\tau}(\psi, b, \Lambda) = F^{\alpha}_{\tau}(\psi, b, \Lambda)/F^{\alpha}_{\tau}(\psi, b, \Lambda)'$ is an irreducible τ-module and $\bar{F}^{\alpha}_{\tau}(\psi, b, \Lambda)$ is integrable if and only if Λ or $-\Lambda$ is a dominant integral weight of $\mathfrak{g}_a$.*
(3) *Let $V \in \vartheta_{\mathrm{fin}}$ be such that $c_0 \neq 0, c_1 = c_2 = \cdots = c_\nu = 0$. Then $V \cong \bar{F}^{\alpha}_{\tau}(\psi, b, \Lambda)$ for some $(\alpha, \Lambda, \psi, b)$.*

Proof. (1) is standard. (3) we have already seen this. (2) We will assume that $c_0 > 0$ as the other case is similar. Let $F = \bar{F}^{\alpha}_{\tau}(\psi, b, \Lambda)$. F is integrable implies that Λ is dominant integral is easy to see and very standard. If Λ is dominant integral then F is integrable is non-trivial and does not follow from our work. But fortunately F has been constructed explicitly by Billig [6] through the use of vertex operator algebras. In particular it is easy to verify that F is integrable if Λ is dominant integral from [6] but one should be familiar with vertex operator algebras notation. □

4. The structure of V with $c_0 = \cdots = c_\nu = 0$

In this section, we assume that $c_0 = \cdots = c_\nu = 0$. By Theorem 2.1,

$$T = \{v \in V \,|\, (\dot{\mathfrak{g}}_+ \otimes A)v = 0\} \neq \{0\}. \tag{4.1}$$

Obviously, T is a $(\mathfrak{b} + \dot{\mathfrak{h}} \otimes A)$-module. Since V is irreducible it follows that

$$h(w) = \Lambda(h)w, \quad \forall h \in \dot{\mathfrak{h}},\ \forall w \in T$$

for some $\Lambda \in P(V)$. If $\Lambda|_{\dot{\mathfrak{h}}} = 0$, then $(\dot{\mathfrak{g}} \otimes A)V = 0$, and V is an irreducible $\mathfrak{b}$-module. In the following discussion, we assume that $\Lambda|_{\dot{\mathfrak{h}}} \neq 0$, so there exists $h_0 \in \dot{\mathfrak{h}}$ such that $\Lambda(h_0) \neq 0$. Let

$$T = \bigoplus_{\mathbf{m} \in \mathbb{Z}^{\nu+1}} T_{\mathbf{m}},$$

where $T_{\mathbf{m}} = \{v \in T \,|\, d_p(v) = (\Lambda(d_p) + m_p)v, 0 \leqslant p \leqslant \nu\}$, $\mathbf{m} = (m_0, \ldots, m_\nu)$.

Considering $h_0 \otimes t^{\mathbf{m}} (\mathbf{m} \in \mathbb{Z}^{\nu+1})$ instead of $t^m k_0 (\mathbf{m} \in \mathbb{Z}^{\nu})$, quite similar to the proof of Lemmas 3.1–3.3, we can deduce that

Lemma 4.1. *For $h_0 \otimes t^{\mathbf{m}} \in \tau$, if there exists a non-zero element w in T such that $h_0 \otimes t^{\mathbf{m}} w = 0$, then $h_0 \otimes t^{\mathbf{m}}$ is locally nilpotent on T and* $\dim T_{\mathbf{n}} > \dim T_{\mathbf{n}+\mathbf{m}}$.

Lemma 4.2. *If both $h_0 \otimes t^{\mathbf{m}}$ and $h_0 \otimes t^{\mathbf{n}}$ are locally nilpotent on T, then so is $h_0 \otimes t^{\mathbf{m}+\mathbf{n}}$. If $h_0 \otimes t^{\mathbf{m}+\mathbf{n}}$ is locally nilpotent, then $h_0 \otimes t^{\mathbf{m}}$* or *$h_0 \otimes t^{\mathbf{n}}$ is locally nilpotent.*

Lemma 4.3. *For any $\mathbf{m} \in \mathbb{Z}^{\nu+1}$, $v \in T \backslash \{0\}$, we have $h_0 \otimes t^{\mathbf{m}} v \neq 0$, and*

$$\dim T_{\mathbf{n}} = \dim T_{\mathbf{m}} = n$$

for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^{\nu+1}$.

Proof. Suppose the lemma is false. By Lemma 4.1, there exists an element in $\{h_0 \otimes t_i^{\pm 1} | 0 \leqslant i \leqslant \nu\}$ which is locally nilpotent. Assume that $h_0 \otimes t_1$ is locally nilpotent, by Lemma 4.2, $\{h_0 \otimes t_1^k | k \geqslant 1\}$ are all locally nilpotent too. Therefore

$$T_{\mathbf{r}+(k,0,\ldots,0)} = 0 \tag{4.2}$$

for all $k \in \mathbf{N}$, where $\mathbf{r}$ satisfies $\dim T_{\mathbf{r}} = \min\{\dim T_{\mathbf{n}} > 0 | \mathbf{n} \in \mathbf{Z}^{\nu+1}\}$. Let v be a non-zero element in $T_{\underline{r}}$, and $\{n_1, n_2, \ldots, n_s\} \subset \mathbf{N}$ such that $n_i \neq n_j$, for $i \neq j$. We say that $\{(e_{-\alpha} \otimes t_1^{n_i})(h_0 \otimes t_1^{-n_i})v | 1 \leqslant i \leqslant s\}(\alpha \in \dot{\Delta}_+, \Lambda(\alpha^\vee) \neq 0, 0 \neq e_\alpha \in \dot{\mathfrak{g}}_\alpha)$ are linearly independent. In fact, assume that

$$\sum_{i=1}^{s} a_i (e_{-\alpha} \otimes t_1^{n_i})(h_0 \otimes t_1^{-n_i})v = 0.$$

Then

$$t_1^{n_j} d_1 e_\alpha \otimes t_1^{-n_j} \left(\sum_{i=1}^{s} a_i (e_{-\alpha} \otimes t_1^{n_i})(h_0 \otimes t_1^{-n_i})v \right) = 0, \quad j = 1, 2, \ldots, s.$$

Therefore

$$\begin{aligned} & t_1^{n_j} d_1 \sum_{i=1}^{s} (e_{-\alpha} \otimes t_1^{n_i})(e_\alpha \otimes t_1^{-n_j})(h_0 \otimes t_1^{-n_i})v \\ & \quad + \sum_{i=1}^{s} a_i(-n_j + n_i)\alpha^\vee \otimes t_1^{n_i} h_0 \otimes t_1^{-n_i} v \\ & \quad + \sum_{i=1}^{s} a_i \alpha^\vee \otimes t_1^{-n_j+n_i}(-n_i) h_0 \otimes t_1^{n_j - n_i} v \\ & \quad + \sum_{i=1}^{s} a_i \alpha^\vee \otimes t_1^{-n_j+n_i}(h_0 \otimes t_1^{-n_i}) t_1^{n_j} d_1 v = 0. \end{aligned}$$

By (4.1)–(4.2), we have

$$a_j(-n_j)\alpha^\vee \cdot h_0 \cdot v = 0.$$

Therefore

$$a_j = 0, \quad j = 1, 2, \ldots, s.$$

Since s can be any positive integer, V_μ is infinite dimensional, where $\mu|_{\mathfrak{h}} = \Lambda - \alpha$, $\mu(d_i) = (\Lambda(d_i) + r_i)$, $i = 0, 1, \ldots, \nu$. This contradicts the assumption that V has finite-dimensional weight spaces. Therefore the lemma holds. $\square$

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis of T_0. Set

$$v_i(\mathbf{m}) = \frac{1}{\Lambda(h_0)} h_0 \otimes t^{\mathbf{m}} \cdot v_i, \quad i = 1, 2, \ldots, m. \tag{4.3}$$

Then $\{v_1(\mathbf{m}), v_2(\mathbf{m}), \ldots, v_m(\mathbf{m})\}$ is a basis of $T_{\mathbf{m}}$. Assume that

$$\begin{aligned}&\frac{1}{\Lambda(h_0)} h_0 \otimes t^{\mathbf{m}}(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_m(\mathbf{n}))\\ &\quad = (v_1(\mathbf{m}+\mathbf{n}), v_2(\mathbf{m}+\mathbf{n}), \ldots, v_m(\mathbf{m}+\mathbf{n}))H_{\mathbf{m},\mathbf{n}}.\end{aligned} \tag{4.4}$$

By Lemma 4.3, $H_{\mathbf{m},\mathbf{n}}$ is invertible. Since $c_0 = c_1 = \cdots = c_\nu = 0$, we have

$$H_{-\mathbf{m},\mathbf{m}} = H_{\mathbf{m},-\mathbf{m}}.$$

Lemma 4.4. (1) *For* $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^{\nu+1}$*, there exist* $\lambda_{\mathbf{m},\mathbf{n}} \in \mathbb{C}$ *and a non-zero element* $v \in T_0$ *such that*

$$(h_0 \otimes t^{\mathbf{m}} h_0 \otimes t^{\mathbf{n}} - \lambda_{\mathbf{m},\mathbf{n}} \Lambda(h_0) h_0 \otimes t^{\mathbf{m}+\mathbf{n}}) \cdot v = 0.$$

(2) $h_0 \otimes t^{\mathbf{m}} h_0 \otimes t^{-\mathbf{m}} - \lambda_{\mathbf{m},-\mathbf{m}} \Lambda(h_0) h_0 \otimes t^{\mathbf{m}}$ *is locally nilpotent on* T.
(3) $H_{\mathbf{m},-\mathbf{m}}$ *does not have different eigenvalues.*

For $\mathbf{m} = (m_0, m_1, \ldots, m_\nu)$, $\mathbf{n} = (n_0, n_1, \ldots, n_\nu) \in \mathbb{Z}^{\nu+1}$, assume that

$$t^{\mathbf{m}} d_a(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_m(\mathbf{n})) = (v_1(\mathbf{m}+\mathbf{n}), v_2(\mathbf{m}+\mathbf{n}), \ldots, v_m(\mathbf{m}+\mathbf{n}))A^{(a)}_{\mathbf{m},\mathbf{n}},$$

where $A^{(a)}_{\mathbf{m},\mathbf{n}} \in \mathbb{C}^{m\times m}$. Since $[t^{\mathbf{m}} d_a, h_0 \otimes t^{\mathbf{n}}] = n_a h_0 \otimes t^{\mathbf{m}+\mathbf{n}}$, $0 \leqslant a \leqslant \nu$, we have

$$A^{(a)}_{\mathbf{m},\mathbf{n}} = H_{\mathbf{m},\mathbf{n}} A^{(a)}_{\mathbf{m},\underline{\mathbf{0}}} + n_a I, \quad 0 \leqslant a \leqslant \nu.$$

Theorem 4.1. $(t^{\mathbf{m}} k_p)T = 0$, $\forall \mathbf{m} \in \mathbb{Z}^{\nu+1}$, $p = 0, 1, \ldots, \nu$.

Proof. It is similar to the proof of (3.5). □

Theorem 4.2. *For all* $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^{\nu+1}$ *and* $h \in \dot{\mathfrak{h}}$*, we have*

$$h \otimes t^{\mathbf{m}}(v_1(\mathbf{n}), v_2(\mathbf{n}), \ldots, v_m(\mathbf{n})) = \Lambda(h)(v_1(\mathbf{m}+\mathbf{n}), v_2(\mathbf{m}+\mathbf{n}), \ldots, v_m(\mathbf{m}+\mathbf{n})).$$

Therefore T *is isomorphic to* $F^\alpha(\psi, b)$*, for some* (α, ψ, b)*, as* $A \oplus \mathrm{Der}\, A$*-modules, where* $F^\alpha(\psi, b) = V(\psi, b) \otimes A$ *is an irreducible* $A \oplus \mathrm{Der}\, A$*-module such that* $V(\psi, b)$ *is a* m*-dimensional irreducible* $\mathrm{gl}_{\nu+1}(\mathbb{C})$*-module, and*

$$\psi(I) = b\, \mathrm{id}_{V(\psi,b)},$$

$$t^{\mathbf{r}} d_p(v \otimes t^{\mathbf{m}}) = (m_p + \alpha_p) v \otimes t^{\mathbf{r}+\mathbf{m}} + \sum_{i=0}^{\nu} r_i \psi(E_{ip}) v \otimes t^{\mathbf{r}+\mathbf{m}}.$$

Let $\Lambda_0 \in \dot{\mathfrak{h}}^*$ be such that $\Lambda_0 \neq 0$, $(L(\Lambda_0), \pi)$ an irreducible highest weight module of $\dot{\mathfrak{g}}$ with the highest weight Λ_0 and the associated highest weight vector v_{Λ_0}, $F^\alpha(\psi, b)$ a

$A \oplus \operatorname{Der} A$-modules defined above. Let

$$F^{\alpha}(\psi, b, \Lambda_0) = L(\Lambda_0) \otimes F^{\alpha}(\psi, b).$$

We define the action of τ on $F^{\alpha}(\psi, b, \Lambda_0)$ by

$$x \otimes t^{\mathbf{r}}(w \otimes v(\mathbf{m})) = (\pi(x)w) \otimes v(\mathbf{m} + \mathbf{r}),$$

$$t^{\mathbf{r}} k_p(w \otimes v(\mathbf{m})) = 0,$$

$$t^{\mathbf{r}} d_p(w \otimes v(\mathbf{m})) = w \otimes t^{\mathbf{r}} d_p v(\mathbf{m}),$$

where $x \in \dot{\mathfrak{g}}$, $w \in L(\Lambda_0)$, $v(\mathbf{m}) = v \otimes t^{\mathbf{m}} \in V(\psi, b) \otimes A$, $\mathbf{r}, \mathbf{m} \in \mathbb{Z}^{\nu+1}$, $0 \leqslant p \leqslant \nu$. Then $F^{\alpha}(\psi, b, \Lambda_0)$ is an irreducible τ-module.

Theorem 4.3. *Let $F^{\alpha}(\psi, b, \Lambda_0)$ be an irreducible τ-module defined above.*

(1) *$F^{\alpha}(\psi, b, \Lambda_0)$ is integrable if and only if Λ_0 is a dominant integral weight of $\dot{\mathfrak{g}}$.*
(2) *Let $V \in \vartheta_{\text{fin}}$ be such that $c_0 = c_1 = \cdots = c_\nu = 0$ and $(\dot{\mathfrak{g}} \otimes A)V \neq 0$. Then $V \cong F^{\alpha}(\psi, b, \Lambda_0)$, for some $(\alpha, \psi, b, \Lambda_0)$.*

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